

STOCHASTIC RECURSIONS: BETWEEN KESTEN'S AND GREY'S ASSUMPTIONS

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ABSTRACT. We study the stochastic recursions $X_n = A_n X_{n-1} + B_n$ and $X_n = \max\{A_n X_{n-1}, B_n\}$, where $(A_n, B_n) \in \mathbb{R} \times \mathbb{R}$ is an i.i.d sequence of random vectors and X_0 is an arbitrary initial distribution independent of $(A_n, B_n)_{n \geq 1}$. The tail behavior of their stationary solutions is well known under the so called Kesten-Grincevičius-Goldie or Grey conditions. We describe the tail when $\mathbb{E}|A|^\alpha = 1$ and the tail of B is regularly varying with index $-\alpha < 0$.

1. INTRODUCTION

1.1. Results. Let $(A_n, B_n) \in \mathbb{R} \times \mathbb{R}$ be a sequence of i.i.d (independent identically distributed) random vectors. Given X_0 independent of $(A_n, B_n)_{n \geq 1}$ we study stochastic recursions

$$(1) \quad X_n = A_n X_{n-1} + B_n, \quad n \geq 1$$

and

$$(2) \quad X_n = \max\{A_n X_{n-1}, B_n\}, \quad n \geq 1.$$

In (2), we assume that $A_n \geq 0$ a.s. Under mild contractivity hypotheses, X_n converges in law to a random variable R satisfying (in distribution)

$$R \stackrel{d}{=} AR + B$$

and

$$R \stackrel{d}{=} \max\{AR, B\},$$

respectively, where (A, B) is a generic element of the sequence $(A_n, B_n)_{n \geq 1}$ and (A, B) , R are independent. We assume that

$$(3) \quad \mathbb{E}|A|^\alpha = 1 \text{ for some } \alpha > 0,$$

$$(4) \quad B \text{ has a regularly varying right tail of order } -\alpha, \quad \mathbb{E}|B|^\alpha = \infty$$

and we describe the right tail of R (Theorems 4.2, 4.4, 4.9). In the most simplified version, our basic result says that if $\mathbb{P}(A \geq 0) = 1$ and there exists $\alpha > 0$ such that $\mathbb{E}A^\alpha = 1$, $\rho = \mathbb{E}A^\alpha \log A < \infty$, $x^\alpha \mathbb{P}(B > x) = L(x)$, where L is a slowly varying function, then (see Section 4 for the rest of assumptions)

$$(5) \quad x^\alpha \mathbb{P}(R > x) \sim \frac{1}{\rho} \int_0^x \frac{L(t)}{t} dt = \frac{\mathbb{E}B^\alpha \mathbf{1}_{0 < B \leq x}}{\alpha \rho} \rightarrow \infty.$$

Here and henceforth, $f(x) \sim g(x)$ means that $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$. Neither affine (1) nor extremal (2) recursions have been considered yet under assumptions (3) and (4) simultaneously and appearance of the function $\int_0^x L(t)t^{-1}dt$ is probably the most interesting phenomenon here. To obtain (5) we prove a renewal theorem that essentially generalizes existing ones, Theorem 3.1.

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For the recursion (1) we do not restrict ourselves to positive A and (5) is obtained in full generality. We first assume that $A \geq 0$ a.s. and then, in Section 4.5, we show how to reduce “signed A ” to “non-negative A ”. The method is quite general and it is applicable beyond our particular assumptions.

Finally, if $A \geq 0$ a.s., we obtain the second order asymptotics in (5) that is, as $x \rightarrow \infty$,

$$(6) \quad \left| x^\alpha \mathbb{P}(R > x) - \frac{1}{\rho} \int_0^x \frac{L(t)}{t} dt \right| = O(\max\{L(x), 1\}),$$

see Theorems 4.2 and 4.8. For (6) in the case of the perpetuity (1) some more regularity of the law of A is assumed.

1.2. History and motivation. $\mathbb{P}(R > x)$ converges to zero when x tends to infinity and a natural problem consists of describing the rate at which this happens. Depending on the assumptions on (A, B) we may obtain light-tailed R (all the moments exist) or a heavy tailed R (certain moments of $|R|$ are infinite). The first case occurs when $\mathbb{P}(|A| \leq 1) = 1$ and B has the moment generating function in some neighborhood of the origin, see Goldie and Grübel [1996], Hitczenko and Wesolowski [2009], Kołodziejek [2016b].

The second one f.e. when $\mathbb{P}(|A| > 1) > 0$ but $\mathbb{E} \log |A| < 0$. In this case the tails of the perpetuity (1) and the maximum of perturbed random walk (2) are comparable (see Enriquez et al. [2009]).¹ Then the tail behaviour of R may be determined by A or B alone, or by both of them. The first case happens when the tail of B is regularly varying with index $-\alpha < 0$ and $\mathbb{E}|A|^\alpha < 1$ and $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then

$$(7) \quad \mathbb{P}(R > x) \sim c\mathbb{P}(B > x),$$

Grey [1994]. When $\mathbb{E}|A|^\alpha = 1$, $\mathbb{E}|B|^\alpha < \infty$ and $\mathbb{E}|A|^a \log^+ |A| < \infty$ then

$$(8) \quad \mathbb{P}(R > x) \sim cx^{-\alpha}$$

and it is A that plays the role. When $\mathbb{E}|A|^a \log^+ |A| = \infty$ an extra slowly varying function l appears in (8) i.e.

$$(9) \quad \mathbb{P}(R > x) \sim cl(x)x^{-\alpha}.$$

(9) was proved by Kevei [2016] for $A \geq 0$ but applying our approach to signed A we may conclude (9) also there.

In view of all that it is natural to go a step further and to ask what happens when at the same time A and B contribute significantly to the tail i.e. $\mathbb{E}|A|^\alpha = 1$, $\mathbb{E}|B|^\alpha = \infty$ and the tail of B is regularly varying at ∞ with index $-\alpha < 0$. Then we may expect that the tail is essentially bigger then that of B and it is what we obtain, see (5).

1.3. Perturbed random walk. There is a somehow related problem, where contributions to asymptotics of some statistic may come from one of two ingredients alone or from both of them. Let $(\xi_n, \eta_n)_{n \geq 1}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η) . Consider the maximum of so-called perturbed random walk, $M_n = \max_{1 \leq k \leq n} \{S_{k-1} + \eta_k\}$, where $(S_n)_{n \geq 1}$ is a random walk with i.i.d. increments ξ_k , $\mathbb{E}\xi_k = 0$ and $\mathbb{E}\xi_k^2 < \infty$, $S_0 = 0$. The aim is to study convergence in distribution of $a_n M_n$ for some suitable chosen deterministic sequence $(a_n)_{n \geq 1}$. There are essentially three distinct cases. In the first case $\mathbb{E}\eta^2 < \infty$, S_n dominates the perturbation and the limit of $a_n M_n$ coincides with the limit of $a_n \max_{1 \leq k \leq n} \{S_{k-1}\}$. In the second one, the tail $\mathbb{P}(\eta > x)$ is regularly varying with index $\gamma \in (-2, 0)$, perturbation η_n dominates the random walk and the limit coincides with the limit of $a_n \max_{1 \leq k \leq n} \{\eta_k\}$. For above see [Hitczenko and Wesolowski, 2011, Theorem 3]. In the most interesting, third case, that is, if

¹For a discussion of the tail asymptotics of perturbed random walks we refer to Araman and Glynn [2006], Palmowski and Zwart [2007].

$\mathbb{P}(\eta > x) \sim cx^{-2}$ for some $c > 0$, both random walk and the perturbation have comparable contributions, see Wang [2014], Iksanov and Pilipenko [2014] along with generalization to functional limit theorems.

1.4. Renewal theorems. To prove the tail asymptotics of R we denote $f(x) = e^{\alpha x} \mathbb{P}(R > e^x)$ and we write a renewal equation for f as in Goldie [1991]. Then

$$f(x) = \int_{\mathbb{R}} \psi(x-z) dH(z)$$

for some ψ , when H is a renewal function. The main difficulty is that ψ is not integrable so the usual approach via the classical renewal theorem doesn't work. There are variants of it when ψ is not necessarily in L_1 Iksanov [2017], but asymptotically equivalent to a monotone function. This doesn't help us either.

We are able to replace $\psi(x)$ by $L(e^x)$ and to show that

$$(10) \quad \int_{\mathbb{R}} L(e^{x-z}) dH(z)$$

is the main term giving the asymptotics but still the behavior of (10) for a generic slowly varying function L remains to be determined. This is done in Section 3, where we prove renewal Theorems 3.1, 3.2 that are interesting on their own.

2. PRELIMINARIES

2.1. Regular variation. A measurable function $f: (0, \infty) \rightarrow (0, \infty)$ is called *regularly varying with index* ρ , $|\rho| < \infty$, if for all $\lambda > 0$,

$$(11) \quad \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho.$$

The class of such functions will be denoted $R(\rho)$. If $f \in R(0)$ then f is called a *slowly varying function*. The class of slowly varying functions plays a fundamental part in the Karamata's theory of regular variability, since if $f \in R(\rho)$, then $f(x) = x^\rho L(x)$ for some $L \in R(0)$. Below, we introduce some basic properties of the class $R(0)$ that, later on, will be essential.

If $L \in R(0)$ is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$, then for any $\delta > 0$ there exists $A = A(\delta) > 1$ such that (Potter's Theorem, see e.g Buraczewski et al. [2016], Appendix B)

$$\frac{L(y)}{L(x)} \leq A \max \left\{ (y/x)^\delta, (y/x)^{-\delta} \right\}, \quad x, y > 0.$$

Assume that $L \in R(0)$ is locally bounded on (X, ∞) for some $X > 0$. Then, for $\alpha > 0$ and $x > X$, one has

$$(12) \quad \int_X^x t^\alpha \frac{L(t)}{t} dt \sim \alpha^{-1} x^\alpha L(x)$$

and this result remains true also for $\alpha = 0$ in the sense that

$$(13) \quad \frac{\int_X^x \frac{L(t)}{t} dt}{L(x)} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

Define $\tilde{L}(x) := \int_X^x \frac{L(t)}{t} dt$. Then, for any $\lambda > 0$,

$$(14) \quad \frac{\tilde{L}(\lambda x) - \tilde{L}(x)}{L(x)} = \int_1^\lambda \frac{L(xt)}{L(x)} \frac{dt}{t} \rightarrow \log \lambda,$$

since the convergence in (11) is locally uniform [Bingham et al., 1989, Theorem 1.5.2]. Moreover, since

$$\frac{\tilde{L}(x)}{L(x)} \left(\frac{\tilde{L}(\lambda x)}{\tilde{L}(x)} - 1 \right) \rightarrow \log \lambda \quad \text{as } x \rightarrow \infty,$$

(13) implies that \tilde{L} is slowly varying. In the theory of regular variation, \tilde{L} is called the de Haan function.

2.2. Renewal theory. Let $(Z_k)_{k \geq 1}$ be the sequence of independent copies of random variable Z with $\mathbb{E}Z > 0$. We write $S_n = Z_1 + \dots + Z_n$, $n \geq 1$ and $S_0 = 0$. The measure defined by

$$H(B) := \sum_{n=0}^{\infty} \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R})$$

is called the *renewal measure* of $(S_n)_{n \geq 1}$. Then, $H(x) := H((-\infty, x])$ is called the *renewal function*. $\mathbb{E}Z > 0$ implies that $H(x)$ is finite for all $x \in \mathbb{R}$.

We say that the distribution of Z is *arithmetic* if its support is contained in $d\mathbb{Z}$ for some $d \in \mathbb{R}$. Equivalently, the distribution of Z is arithmetic if and only if there exists $0 \neq t \in \mathbb{R}$ such that $f_Z(t) = 1$, where f_Z is the characteristic function of the distribution of Z . The law of Z is *strongly non-lattice* if the Cramer's condition is satisfied, that is, $\limsup_{|t| \rightarrow \infty} |f_Z(t)| < 1$.

A fundamental result of renewal theory is the Blackwell's theorem: if the distribution of Z is non-arithmetic, then for any $t > 0$,

$$H(x+t) - H(x) \rightarrow \frac{t}{\mathbb{E}Z} \quad \text{as } x \rightarrow \infty.$$

Under additional assumptions we know more about the asymptotic behaviour of H (see Stone [1965]). If for some $r > 0$ one has $\mathbb{P}(Z \leq x) = o(e^{rx})$ as $x \rightarrow -\infty$, then there is some $r_1 > 0$ such that

$$(15) \quad H(x) = o(e^{r_1 x}) \quad \text{as } x \rightarrow -\infty.$$

Exact asymptotics of $H(x)$ as $x \rightarrow -\infty$ in the presence of $\alpha > 0$ such that $\mathbb{E}e^{-\alpha X} = 1$ are given in Kołodziejek [2016a].

If for some $r > 0$, $\mathbb{P}(Z > x) = o(e^{-rx})$ as $x \rightarrow \infty$ and the distribution of Z is strongly non-lattice, then

$$(16) \quad H(x) = \frac{x}{\mathbb{E}Z} + \frac{\mathbb{E}Z^2}{2(\mathbb{E}Z)^2} + o(e^{-rx}) \quad \text{as } x \rightarrow \infty.$$

Note that in the non-arithmetic case, since $H(x+t) - H(x)$ is convergent as $x \rightarrow \infty$ we have $C = \sup_x (H(x+1) - H(x)) < \infty$ and so

$$(17) \quad H(x+h) - H(x) \leq [h] C \leq \alpha h + \beta$$

for some positive α, β and any $h > 0$. In renewal theory it is usually easier first to consider non-negative Z , and then to extend some argument to arbitrary Z using the following approach. Let $N = \inf\{n \in \mathbb{N} : S_n > 0\}$ be the first ladder epoch of $(S_n)_{n \geq 1}$. We define a measure by

$$V(B) := \mathbb{E} \left(\sum_{n=0}^{N-1} \mathbf{1}_{S_n \in B} \right), \quad B \in \mathcal{B}(\mathbb{R}).$$

V is a finite measure with support contained in $(-\infty, 0]$ and total mass equal $\mathbb{E}N$. Since $(S_n)_{n \geq 1}$ has a positive drift, $\mathbb{E}N$ is finite. Let $Z_1^> \stackrel{d}{=} S_N$ be the first ladder height of $(S_n)_{n \geq 1}$ and consider an i.i.d. sequence $(Z_n^>)_{n \geq 1}$. Then, it can be shown that

$$H = V * H^>,$$

where $H^>$ is the renewal measure of $(S_n^>)_{n \geq 1}$ and $S_n^> = \sum_{k=1}^n Z_k^>$ ([Blackwell, 1953, Theorem 2], see also [Alsmeyer, 2015, Lemma 2.63] for more general formulation).

For a part of our results we need a better control of $H(x+h) - H(x)$ in terms of h than in (17). Something in the spirit of

$$H(x+h) - H(x) \leq ch^\beta, \quad x \geq 0, h > 0$$

for some $\beta > 0$. Observe that with $C_n = \sup_x \{H(x + 1/n) - H(x)\} < \infty$ we have

$$H(x + h) - H(x) \leq C_n \frac{\lceil nh \rceil}{n}$$

thus (19) holds for all x and $h > 1/n$ with $\beta = 1$. Hence, we have to investigate the case of small h only. We have the following statement.

Lemma 2.1 *Assume that $\mathbb{P}(Z > x) = o(e^{-rx})$ as $x \rightarrow \infty$ for some $r > 0$ and that the law of Z is strongly non-lattice. If there exists $\beta > 0$ such that*

$$(18) \quad \limsup_{h \rightarrow 0^+} \sup_{a \geq 0} h^{-\beta} \mathbb{P}(a < Z \leq a + h) < \infty,$$

then there exists $\tilde{\beta} > 0$ and $c > 0$ such that for $x \geq 0$ and $h \in \mathbb{R}_+$,

$$(19) \quad H(x + h) - H(x) \leq c \max\{h^{\tilde{\beta}}, h\}.$$

Remark 2.2 *Notice that (18) is satisfied when the law of Z has density in L^p for some $1 < p \leq \infty$.*

Proof. Assume first that $Z \geq 0$ with probability 1 and let F be the cumulative distribution function of Z . From condition (18) we infer that there exists $\beta, c, \varepsilon > 0$ such that for any $a \geq 0$ and any $h \in (0, \varepsilon]$ one has $F(a + h) - F(a) = \mathbb{P}(a < Z \leq a + h) \leq ch^\beta$. Since $H(x) = \mathbf{1}_{x \geq 0} + H * F(x)$ we have for any $x \geq 0$ and $h \in (0, \varepsilon]$,

$$\begin{aligned} H(x + h) - H(x) &= \int_{[0, x]} (F(x - z + h) - F(x - z)) dH(z) + \int_{(x, x+h]} F(x + h - z) dH(z) \\ &\leq ch^\beta H(x) + F(h)(H(x + h) - H(x)) \end{aligned}$$

and thus

$$H(x + h) - H(x) \leq \tilde{c} h^\beta H(x)$$

provided $F(\varepsilon) < 1$.

Let now Z be arbitrary and let S_N be the first ladder height of $(S_n)_{n \geq 1}$. Since $\mathbb{E}N < \infty$ and

$$\begin{aligned} \mathbb{P}(a < S_N \leq a + h) &= \sum_{n=1}^{\infty} \mathbb{P}(a < S_n \leq a + h, S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(a - S_{n-1} < Z_n \leq a - S_{n-1} + h, N \geq n) \leq ch^\beta \sum_{n=1}^{\infty} \mathbb{P}(N \geq n), \end{aligned}$$

condition (18) implies that

$$\limsup_{h \rightarrow 0^+} \sup_{a \geq 0} h^{-\beta} \mathbb{P}(a < S_N \leq a + h) < \infty.$$

Thus, using factorization $H = V * H^>$ we obtain for $x \geq 0$ and $h \in (0, \varepsilon]$,

$$H(x + h) - H(x) = \int_{(-\infty, 0]} H^>((x - t, x - t + h]) V(dt) \leq ch^\beta \int_{(-\infty, 0]} H^>(x - t) V(dt) = ch^\beta H(x).$$

For $0 \leq x \leq h^{-\delta}$ with $\delta < \beta$ this implies that

$$H(x + h) - H(x) \leq Ch^\beta(1 + x) \leq \tilde{C}h^{\beta-\delta}.$$

On the other hand, for $x > h^{-\delta}$ and $r > 0$ we have

$$e^{-rx} \leq e^{-rh^{-\delta}} \leq h.$$

and so the conclusion follows by (16), since then

$$H(x + h) - H(x) = \frac{h}{\mathbb{E}Z} + o(e^{-rx}).$$

□

3. RENEWAL THEOREMS

A function $f: \mathbb{R} \rightarrow \mathbb{R}_+$ is called *directly Riemann integrable* on \mathbb{R} (dRi) if for any $h > 0$,

$$(20) \quad \sum_{n \in \mathbb{Z}} \sup_{(n-1)h \leq y < nh} f(y) < \infty$$

and

$$\lim_{h \rightarrow 0^+} h \left(\sum_{n \in \mathbb{Z}} \sup_{(n-1)h \leq y < nh} f(y) - \sum_{n \in \mathbb{Z}} \inf_{(n-1)h \leq y < nh} f(y) \right) = 0.$$

If f is locally bounded and a.e. continuous on \mathbb{R} , then an elementary calculation shows that (20) with $h = 1$ implies direct integrability of f . For directly Riemann integrable function f , we have the following *Key Renewal Theorem*:

$$\int_{\mathbb{R}} f(x-z) dH(z) \rightarrow \frac{1}{\mathbb{E}Z} \int_{\mathbb{R}} f(t) dt.$$

There are many variants of this theorem, when f is not necessarily L_1 - see [Iksanov, 2017, Section 6.2.3]. Such results are usually obtained by additional requirement that f is (ultimately) monotone or f is asymptotically equivalent to a monotone function.

Here we obtain a renewal result that is essentially stronger: an asymptotic of

$$\int_{[0,x]} L(e^{x-z}) dH(z)$$

for a slowly varying function L , Theorem 3.2. Such a function may exhibit infinite oscillations, so in general it is not asymptotically equivalent to a monotone function. The following Theorem is an intermediate step: we assume monotonicity of $x^{-\alpha}L(x)$.

Theorem 3.1 *Assume that $0 < \mathbb{E}Z < \infty$, the law of Z is non-arithmetic and $\mathbb{P}(Z \leq x) = o(e^{rx})$ as $x \rightarrow -\infty$. Assume further that there is a random variable B and a slowly varying function L such that $\mathbb{P}(B > x) = x^{-\alpha}L(x)$. Let $\tilde{L}(x) = \int_0^x \frac{L(t)}{t} dt$. Then,*

$$(21) \quad \int_{(0,x]} L(e^{x-z}) dH(z) \sim \int_{\mathbb{R}} L(e^{x-z}) dH(z) \sim \frac{1}{\mathbb{E}Z} \tilde{L}(e^x).$$

Moreover, if g is such that g, g' are bounded and $\text{supp } g \subset [1, \infty)$, then

$$(22) \quad \lim_{x \rightarrow \infty} \tilde{L}(e^x)^{-1} \int_{\mathbb{R}} e^{\alpha(x-z)} \mathbb{E}g(e^{z-x}B) dH(z) = \alpha \int_{\mathbb{R}} g(r) r^{-\alpha-1} dr.$$

Assume additionally that the distribution of Z is strongly non-lattice and that $\mathbb{E}e^{\varepsilon Z} < \infty$ for some $\varepsilon > 0$. Then

$$(23) \quad \int_{\mathbb{R}} L(e^{x-z}) dH(z) = \frac{1}{\mathbb{E}Z} \tilde{L}(e^x) + O(L(e^x)).$$

All our further results are based on Theorem 3.1. The proof of it is, however, quite long and so it is postponed to Section 5.

Theorem 3.2 *Assume that $0 < \mathbb{E}Z < \infty$ and the law of Z is non-arithmetic. For any slowly varying function L such that $\tilde{L}(x) = \int_0^x \frac{L(t)}{t} dt$ is finite for any $x > 0$ and $\tilde{L}(x) \rightarrow \infty$ as $x \rightarrow \infty$, one has*

$$\int_{(0,x]} L(e^{x-z}) dH(z) \sim \frac{\tilde{L}(e^x)}{\mathbb{E}Z}.$$

Proof. It is known that any slowly varying function is asymptotically equivalent to a smooth one, say L_0 ([Bingham et al., 1989, Theorem 1.8.2]), such that

$$(24) \quad \lim_{x \rightarrow \infty} \frac{xL'_0(x)}{L_0(x)} = 0.$$

Moreover, observe that $\tilde{L}_0(x) := \int_0^x L_0(t)t^{-1}dt \sim \tilde{L}(x)$, provided $L_0(t)t^{-1}$ is integrable on a right neighbourhood of 0.

Thus, for x large enough, say $x \geq e^N$, one has

$$1 - \varepsilon \leq \frac{L(x)}{L_0(x)} \leq 1 + \varepsilon.$$

This implies that

$$(1 - \varepsilon) \int_{(0, x-N]} L_0(e^{x-z})dH(z) \leq \int_{(0, x-N]} L(e^{x-z})dH(z) \leq (1 + \varepsilon) \int_{(0, x-N]} L_0(e^{x-z})dH(z).$$

On the other hand, since the function $Q(x) = L(e^x)\mathbf{1}_{x < N}$ is compactly supported, it is dRi. Thus, by the Key Renewal Theorem, we have

$$\int_{(x-N, x]} L(e^{x-z})dH(z) = \int_{(0, x]} Q(x-z)dH(z) \rightarrow \frac{1}{\mathbb{E}Z} \int_0^\infty Q(t)dt = \frac{\tilde{L}(e^N)}{\mathbb{E}Z} < \infty.$$

It remains to show that $\int_{(0, x-N]} L_0(e^{x-z})dH(z) \sim \frac{\tilde{L}_0(e^x)}{\mathbb{E}Z} \sim \frac{\tilde{L}(e^x)}{\mathbb{E}Z}$. For this it is enough to prove that $\frac{L_0(x)}{x^\alpha}$ is a probability distribution and to apply Theorem 3.1. Observe, that there exists $\alpha > 0$ such that $\frac{L_0(x)}{x^\alpha}$ is decreasing. Indeed, by (24),

$$\frac{d}{dx} \frac{L_0(x)}{x^\alpha} = \frac{L_0(x)}{x^{1+\alpha}} \left(\frac{xL'_0(x)}{L_0(x)} - \alpha \right) < 0$$

for α large enough.

Since L_0 may be taken arbitrary on the set $(0, X]$, $\frac{L_0(x)}{x^\alpha} =: \mathbb{P}(B > x)$ defines a probability distribution and the conclusion follows. \square

4. TAIL ASYMPTOTICS

4.1. Notation and assumptions. Throughout the paper, \log stands for the natural logarithm. We are going to write a_+ and a_- for $\max\{a, 0\}$ and $\max\{-a, 0\}$ respectively, and we adopt the usual convention that $0^\alpha \log 0 = 0$.

For any $n \geq 1$ we write $\Pi_n = A_1 \cdot \dots \cdot A_n$ and $\Pi_0 = 1$.

Our standing assumptions are:

- (A-1) $\mathbb{P}(A \geq 0) = 1$, $\mathbb{E} \log A < 0$,
- (A-2) There exists $\alpha > 0$ such that $\mathbb{E}A^\alpha = 1$, $\mathbb{E}A^\alpha \log A < \infty$,
- (A-3) the law of $\log A$ given $A > 0$ is non-arithmetic,
- (B-1) $L(x) := x^\alpha \mathbb{P}(B > x) \in R(0)$,
- (B-2) $\mathbb{E}B_+^\alpha = \infty$,
- (B-3) $\mathbb{E}B_-^{\alpha-\eta} < \infty$ for any $\eta \in (0, \alpha)$.

Note that under (A-2)

$$\rho = \mathbb{E}A^\alpha \log A$$

is strictly positive. Indeed, consider $f(\beta) := \mathbb{E}A^\beta$. Since $f(0) = 1 = f(\alpha)$, f is convex, we have $f'(\alpha) = \rho > 0$.

Define

$$\tilde{L}(x) := \int_0^x \frac{L(t)}{t} dt.$$

As an easy consequence of (12) we obtain

Proposition 4.1 *Suppose that (B-1) is satisfied. Then*

$$\mathbb{E}B_+^\alpha \mathbf{1}_{B \leq x} = \alpha \tilde{L}(x)$$

and for $r > 0$,

$$\mathbb{E}B_+^{\alpha+r} \mathbf{1}_{B \leq x} = (\alpha + r) \int_0^x t^{\alpha+r-1} \mathbb{P}(B > t) dt \sim \frac{\alpha + r}{r} x^r L(x).$$

Under (B-1), condition (B-2) implies that $\tilde{L}(x) \rightarrow \infty$ as $x \rightarrow \infty$.

In this chapter the previous results in the renewal theory will be applied to the random variable Z with the law defined by

$$(25) \quad \mathbb{P}(Z \in \cdot) = \mathbb{E}A^\alpha \mathbf{1}_{\log A \in \cdot}.$$

4.2. Perturbed random walk. In the following section we consider the supremum of the perturbed multiplicative random walk

$$R = \sup_{n \geq 1} \{\Pi_{n-1} B_n\},$$

where $(A_n, B_n)_{n \geq 1}$ are independent copies of (A, B) . It is clear that R satisfies the maximum equation

$$R \stackrel{d}{=} \max\{AR, B\}, \quad R \text{ and } (A, B) \text{ are independent.}$$

Since $\mathbb{E}R_+^\beta \leq \mathbb{E}B_+^\beta + \mathbb{E}A^\beta \mathbb{E}R_+^\beta$, under (A-2) and (B-1) with $\beta < \alpha$, we have $\mathbb{E}A^\beta < 1$ and $\mathbb{E}B_+^\beta < \infty$. Thus R has finite moments up to α . The main theorem of this section is

Theorem 4.2 *Assume (A-1)-(A-3) and (B-1)-(B-2). If $\mathbb{E}A^\eta B_+^{\alpha-\eta} < \infty$ for some $\eta \in (0, \alpha)$, then*

$$x^\alpha \mathbb{P}(R > x) \sim \frac{\tilde{L}(x)}{\rho}.$$

If additionally $\mathbb{E}A^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$ and the distribution of Z defined by (25) is strongly non-lattice, then

$$(26) \quad x^\alpha \mathbb{P}(R > x) = \frac{\tilde{L}(x)}{\rho} - \frac{\mathbb{E} \min\{AR, B\}_+^\alpha}{\alpha \rho} + O(L(x)).$$

Remark 4.3 *We say that the law μ is spread-out if there exists $n \in \mathbb{N}$ such that n -th convolution μ^{*n} has non-zero absolutely continuous part. Notice that if the law of $\log A$ is spread-out then the law of Z is spread-out and so it is strongly non-lattice. If the law of A has a nontrivial absolutely continuous component then the same holds for $\log A$ implying that the law of Z is strongly non-lattice and we have (26).*

Assumption $\mathbb{E}A^{\alpha+\varepsilon} < \infty$ implies through Hölder inequality the existence of $\eta \in (0, \alpha)$ such that $\mathbb{E}A^\eta B_+^{\alpha-\eta} < \infty$. There is also a weaker condition formulated in Lemma 6.1.

By (26) we have for any $\lambda \geq 1$,

$$(\lambda x)^\alpha \mathbb{P}(R > \lambda x) - x^\alpha \mathbb{P}(R > x) = O(L(x)), \quad \text{as } x \rightarrow \infty,$$

which means that $x \mapsto x^\alpha \mathbb{P}(R > x) \in O\Pi_L$ (see [Bingham et al., 1989, Chapter 3]).

Proof. Let $f(x) = e^{\alpha x} \mathbb{P}(R > e^x)$ and $\psi(x) = e^{\alpha x} (\mathbb{P}(R > e^x) - \mathbb{P}(AR > e^x))$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}$ be a Borel function. If $(Z_k)_k$ is an i.i.d. sequence with the law (25), then

$$\mathbb{E}g(Z_1, \dots, Z_n) = \mathbb{E}\Pi_n^\alpha g(\log A_1, \dots, \log A_n),$$

where $(A_k)_k$ are i.i.d. In particular,

$$(27) \quad \mathbb{E}Z = \mathbb{E}A^\alpha \log A \in (0, \infty).$$

Then,

$$(28) \quad f(x) = \psi(x) + \mathbb{E}A^\alpha f(x - \log A) = \psi(x) + \mathbb{E}f(x - Z).$$

Iterating (28) we obtain

$$f(x) = \sum_{k=0}^{n-1} \mathbb{E}\psi(x - S_k) + \mathbb{E}f(x - S_n),$$

where $S_n = Z_1 + \dots + Z_n$, $S_0 = 0$. Clearly, if the law of $\log A$ given $A > 0$ is non-arithmetic under \mathbb{P} , then the law of Z is non-arithmetic as well.

By (27), the random walk $(S_n)_{n \geq 1}$ has positive drift, thus $S_n \rightarrow \infty$ with probability 1 as $n \rightarrow \infty$. Moreover, $\lim_{x \rightarrow -\infty} f(x) = 0$ and we conclude that $\mathbb{E}f(x - S_n) \rightarrow 0$ as $n \rightarrow \infty$ and so

$$f(x) = \sum_{k=1}^{\infty} \mathbb{E}\psi(x - S_k) = \int_{\mathbb{R}} \psi(x - z) dH(z),$$

where H is the renewal function of $(S_n)_{n \geq 1}$.

In our case ψ is not dRi (it is not even in L_1), so the Key Renewal Theorem is not applicable. Instead, we consider $\psi_B(x) = e^{\alpha x} \mathbb{P}(B > e^x) = L(e^x)$ and define $\psi_0 = \psi - \psi_B$. First we will show that $\int_{\mathbb{R}} \psi_0(x - z) dH(z)$ is convergent as $x \rightarrow \infty$ to a finite limit. Therefore, $\int_{\mathbb{R}} \psi_B(x - z) dH(z)$ will constitute the main part (see Theorem 3.1). Indeed, $\psi_0(x) = -e^{\alpha x} \mathbb{P}(\min\{AR, B\} > e^x)$ and

$$(29) \quad \int_{\mathbb{R}} e^{\alpha(x-z)} \mathbb{P}(\min\{AR, B\} > e^{x-z}) dH(z) = \mathbb{E} \int_{(x-D, \infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{\min\{AR, B\} > 0},$$

where $D = \log \min\{AR, B\}$. Integrating by parts and changing the variable $t = z - x + D$, we obtain

$$- \int_{\mathbb{R}} \psi_0(x - z) dH(z) = \alpha \mathbb{E} \min\{AR, B\}_+^\alpha \int_0^\infty e^{-\alpha t} (H(x - D + t) - H(x - D)) dt.$$

By (17), we may take the limit as $x \rightarrow \infty$ inside the integral. Thus, by the Blackwell Theorem we get

$$- \int_{\mathbb{R}} \psi_0(x - z) dH(z) \rightarrow \mathbb{E} \min\{AR, B\}_+^\alpha \int_0^\infty \alpha e^{-\alpha t} \frac{t}{\mathbb{E}Z} dt = \frac{\mathbb{E} \min\{AR, B\}_+^\alpha}{\alpha \rho}.$$

For the main part, we have

$$\int_{\mathbb{R}} \psi_B(x - z) dH(z) = \int_{\mathbb{R}} L(e^{x-z}) dH(z).$$

Thus, we have to check that the assumptions of Theorem 3.1 are satisfied. We already know that the expectation of Z is strictly positive and finite. Moreover, the law of Z is non-arithmetic. Further, we have $\mathbb{P}(Z \leq x) = \mathbb{E}A^\alpha \mathbf{1}_{\log A \leq x} \leq e^{\alpha x}$ for any $x \in \mathbb{R}$. Finally, observe that $\mathbb{E}e^{\varepsilon Z} = \mathbb{E}A^{\alpha+\varepsilon} < \infty$.

In terms of second order asymptotics, so far we have shown that

$$(30) \quad e^{\alpha x} \mathbb{P}(R > e^x) = \frac{\tilde{L}(e^x)}{\rho} - \frac{\mathbb{E} \min\{AR, B\}_+^\alpha}{\alpha \rho} + o(1) + O(L(e^x)),$$

where

$$o(1) = \int_{\mathbb{R}} \psi_0(x - z) dH(z) + \frac{\mathbb{E} \min\{AR, B\}_+^\alpha}{\alpha \rho} =: K(x)$$

is the error term coming from the integral of ψ_0 . However, L may be decreasing to 0 (f.e. $L(t) = 1/\log(t)$) and we want to be more precise here. We will show that for some $\delta > 0$,

$$K(x) = o(e^{-\delta x}).$$

and in such case we may drop $o(1)$ in (30). Let us first note that if $\mathbb{E}A^{\alpha+\varepsilon} < \infty$, then $\mathbb{E}\min\{AR, B\}_+^{\alpha+\delta} < \infty$ for $\delta < \frac{\alpha\varepsilon}{\alpha+\varepsilon}$. Indeed, we have

$$\mathbb{E}B^{\alpha+\delta}\mathbf{1}_{0 < B \leq AR} \leq \mathbb{E}R^\eta \mathbb{E}B_+^{\alpha+\delta-\eta} A^\eta \leq \mathbb{E}R^\eta \left(\mathbb{E}B_+^{q(\alpha+\delta-\eta)} \right)^{1/q} (\mathbb{E}A^{\eta p})^{1/p},$$

where $p^{-1} + q^{-1} = 1$ and $\eta > 0$. The right hand side is finite for $\eta \in (\delta \frac{\alpha+\varepsilon}{\varepsilon}, \alpha)$ with $p = \frac{\alpha+\varepsilon}{\eta}$. Analogously we show that $\mathbb{E}(AR)_+^{\alpha+\delta} \mathbf{1}_{B > AR} < \infty$. We write (recall that $D = \log \min\{AR, B\}$)

$$\begin{aligned} K(x) &= -\alpha \mathbb{E} \min\{AR, B\}_+^\alpha \int_0^\infty e^{-\alpha t} \left(H(x - D + t) - H(x - D) - \frac{t}{\mathbb{E}Z} \right) dt \mathbf{1}_{D \leq x} \\ &\quad - \alpha \mathbb{E} \min\{AR, B\}_+^\alpha \int_0^\infty e^{-\alpha t} (H(x - D + t) - H(x - D)) dt \mathbf{1}_{D > x} \\ &\quad + \frac{\mathbb{E} \min\{AR, B\}_+^\alpha \mathbf{1}_{D > x}}{\alpha \mathbb{E}Z} = K_1 + K_2 + K_3. \end{aligned}$$

We have

$$|K_2 + K_3| \leq C \mathbb{E} \min\{AR, B\}_+^\alpha \mathbf{1}_{\min\{AR, B\} > e^x} \leq ce^{-\delta x} \min\{AR, B\}_+^{\alpha+\delta}.$$

Moreover, under our setup we know that for $R(x) = H(x) - \frac{x}{\mathbb{E}Z}$ one has $R(x) - \frac{\mathbb{E}Z^2}{2(\mathbb{E}Z)^2} = o(e^{-rx})$ as $x \rightarrow \infty$ and thus $|R(x) - \frac{\mathbb{E}Z^2}{2(\mathbb{E}Z)^2}| \leq Ce^{-rx}$ for some $C > 0$ and $0 < r < \delta$ and all $x \geq 0$. Then

$$|K_1| \leq \alpha \mathbb{E} \min\{AR, B\}_+^\alpha \int_0^\infty e^{-\alpha t} (|R(x - D + t)| + |R(x - D)|) dt \mathbf{1}_{D \leq x} \leq \tilde{C} e^{-rx} \mathbb{E} \min\{AR, B\}_+^{\alpha+r}$$

and the conclusion follows. \square

4.3. Perpetuity - first order asymptotics. In this section we consider the following random affine equation

$$(31) \quad R \stackrel{d}{=} AR + B, \quad \text{where } (A, B) \text{ and } R \text{ are independent.}$$

Given (A, B) that satisfies (A-1) and (B-1), there is a unique solution R to (31). We are going to describe the first order asymptotics of R under the same assumptions as in Theorem 4.2. The proof, however, is not that simple because in principle ψ_0 may not be dRi. So one may proceed as in Goldie [1991]: first show that ψ_0 is in L_1 , then apply a regularization procedure and, finally, deregularize solution using some Tauberian argument. But in this case even to prove that $\psi_0 \in L_1$ constitutes already a challenge and the rest is quite elaborated. So it seems that a different approach, introduced in Buraczewski et al. [2009] may be a way out. Instead of finding the asymptotics of $\mathbb{P}(R > x)$ we look for the asymptotics of $\mathbb{E}g(R/x)$, where g is a Hölder function and $\text{supp } g \subset [1, \infty)$. The advantage of such approach is that the analog of function ψ_0 is easily shown to be dRi (see Proposition 4.6). Moreover, the asymptotics of $\mathbb{P}(R > x)$ follows simply from the asymptotics of $\mathbb{E}g(R/x)$.

However, for the second order asymptotics of $\mathbb{P}(R > x)$ ‘‘Hölder function’’ approach doesn’t work. This problem will be treated in the next Section.

Theorem 4.4 *Assume (A-1)-(A-3) and (B-1)-(B-2). If $\mathbb{E}|B|^{\alpha-\varepsilon} A^\varepsilon < \infty$ for some $0 < \varepsilon \leq 1/2$ with $\varepsilon < \alpha/2$, then*

$$x^\alpha \mathbb{P}(R > x) \sim \frac{\tilde{L}(x)}{\rho}.$$

For $0 < \varepsilon \leq 1/2$, we define H^ε to be the set of bounded functions g satisfying

$$\|g\|_\varepsilon = \sup_{x, y \in \mathbb{R}} \frac{|g(x) - g(y)|}{|x - y|^{2\varepsilon}} < \infty.$$

Theorem 4.4 is an immediate consequence of the following one

Theorem 4.5 *Suppose that the assumptions of Theorem 4.4 are satisfied. Suppose that $g \in H^\varepsilon$, $\text{supp } g \subset [1, \infty)$, g' exists and is bounded. Then*

$$(32) \quad x^\alpha \mathbb{E}g(x^{-1}R) \sim \frac{\alpha}{\rho} \int_{\mathbb{R}} g(r)r^{-\alpha-1}dr \tilde{L}(x).$$

To prove Theorem 4.4 we apply Theorem 4.5 to bounded functions g such that g' is compactly supported in $[1, \infty)$. They are clearly in H^ε because for $|x - y| \leq C$ we have $|x - y| \leq C^{1-2\varepsilon}|x - y|^{2\varepsilon}$.

First we prove Theorem 4.4 and the rest of the section will be devoted to the proof of Theorem 4.5.

Proof of Theorem 4.4. It is enough to prove that for a $\xi > 1$

$$(33) \quad \lim_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{P}(R > x\xi) = \xi^{-\alpha}.$$

Let $\xi > 1$ and $\eta > 0$ be such that $\xi - \eta > 1$. Let g_1 be a C^1 function such that $0 \leq g_1 \leq 1$ and

$$(34) \quad g_1(x) = \begin{cases} 0 & \text{if } x \leq \xi - \eta \\ 1 & \text{if } x \geq \xi \end{cases},$$

Let $g_2(x) = g_1(x - \eta)$.

Then g_1, g_2 satisfy the assumptions of Theorem 4.5 because $g'_1(x), g'_2(x) = 0$ for $x \leq \xi - \eta$ and $x \geq \xi + \eta$. We have

$$\begin{aligned} I_2 &:= \lim_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{E}g_2(x^{-1}R) \\ &\leq \liminf_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{P}(R > x\xi) \leq \limsup_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{P}(R > x\xi) \\ &\leq \lim_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{E}g_1(x^{-1}R) =: I_1. \end{aligned}$$

Moreover, for every $\eta < 1 - \varepsilon$,

$$\begin{aligned} |I_1 - I_2| &\leq \frac{\alpha}{\rho} \int_0^\infty |g_1(r) - g_2(r)| r^{-\alpha-1} dr \\ &\leq \frac{\alpha}{\rho} \int_{\xi-\eta}^{\xi+\eta} r^{-\alpha-1} dr \leq 2\alpha\eta/\rho, \end{aligned}$$

which proves that

$$\lim_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{P}(R > x\xi) \text{ exists}$$

and for every η

$$\left| \lim_{x \rightarrow \infty} x^\alpha \tilde{L}(x)^{-1} \mathbb{P}(R > x\xi) - \xi^{-\alpha} \right| \leq 3\alpha\eta/\rho.$$

Hence the conclusion follows. \square

Proof of Theorem 4.5. Now we are going to prove Theorem 4.5. We assume that $\text{supp } g \subset [1, \infty)$ and we write

$$f(x) = e^{\alpha x} \mathbb{E}g(e^{-x}R).$$

Let

$$\psi_0(x) = e^{\alpha x} \mathbb{E}((g(e^{-x}(AR + B)) - g(e^{-x}AR) - g(e^{-x}B)),$$

and

$$\psi_B(x) = e^{\alpha x} \mathbb{E}g(e^{-x}B) \leq e^{\alpha x} \mathbb{P}(B > e^x) = L(e^x).$$

Proceeding similarly as in the proof of Theorem 4.2, we obtain

$$f(x) = \int_{\mathbb{R}} \psi_0(x-z) dH(z) + \int_{\mathbb{R}} \psi_B(x-z) dH(z),$$

where H is the renewal function of $(S_n)_{n \geq 1}$ and $S_n = Z_1 + \dots + Z_n$, where the distribution of Z_i is defined in (25). By Theorem 4.6, ψ_0 is directly Riemann integrable and so

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}} \psi(x-z) dH(z) = (\mathbb{E}Z)^{-1} \int_{\mathbb{R}} \psi(x) dx < \infty.$$

Therefore, Theorem 4.5 follows from Theorem 3.1 similarly as in the proof of Theorem 4.2. \square

In the next proposition we do not need to assume that $A \geq 0$ with probability 1 nor that R is the solution of the equation $R \stackrel{d}{=} AR + B$. We require only that the moments of $|R|$ of order strictly smaller than α are finite, which is satisfied in our framework.

Proposition 4.6 *Suppose that A, B, R are real valued random variables and (A, B) is independent of R . Assume further that $0 < \varepsilon < \frac{\alpha}{2}$, $\varepsilon \leq \frac{1}{2}$ and $\mathbb{E}|A|^\alpha < \infty$, $\mathbb{E}|B|^{\alpha-\varepsilon}|A|^\varepsilon < \infty$, $\mathbb{E}|R|^\beta < \infty$ for every $\beta < \alpha$. Then for every $g \in H^\varepsilon$ such that $0 \notin \text{supp } g$ the function*

$$\psi_0(x) = e^{\alpha x} \mathbb{E}(g(e^{-x}(AR + B)) - g(e^{-x}AR) - g(e^{-x}B))$$

is directly Riemann integrable.

Proof. Since ψ is continuous it is enough to prove that

$$(35) \quad \sum_{n \in \mathbb{Z}} \sup_{\{x: n \leq x < n+1\}} |\psi(x)| < \infty.$$

For $x, y \in \mathbb{R}$ we have

$$|g(x+y) - g(x) - g(y)| \leq |g(x+y) - g(x)| + |g(y) - g(0)| \leq 2\|g\|_\varepsilon |y|^{2\varepsilon}.$$

Interchanging the roles of x and y , we arrive at

$$|g(x+y) - g(x) - g(y)| \leq 2\|g\|_\varepsilon |y|^\varepsilon |x|^\varepsilon$$

Moreover, $|g(x+y) - g(x) - g(y)| = 0$ if $|x| + |y| < \eta$, where $\text{supp } g \subset \{x : |x| > \eta\}$. Thus,

$$|\psi(x)| \leq Ce^{\alpha x} \mathbb{E}|e^{-x}B|^\varepsilon |e^{-x}AR|^\varepsilon \mathbf{1}_{|AR|+|B| \geq e^x \eta}.$$

We have

$$\sup_{\{x: n \leq x < n+1\}} |\psi(x)| \leq Ce^{(\alpha-2\varepsilon)(n+1)} \mathbb{E}|B|^\varepsilon |AR|^\varepsilon \mathbf{1}_{|AR|+|B| \geq e^n \eta}$$

and

$$\sum_{n \in \mathbb{Z}} \sup_{\{x: n \leq x < n+1\}} |\psi(x)| \leq C\mathbb{E} \sum_{n=-\infty}^{n_0} e^{(\alpha-2\varepsilon)(n+1)} \mathbb{E}|B|^\varepsilon |AR|^\varepsilon$$

where

$$n_0 = \lfloor \log(|AR| + |B|) - \eta \rfloor.$$

Hence there is a constant $C = C(\eta, \alpha, \varepsilon)$ such that

$$\begin{aligned} \sum_n \sup_{\{x: n \leq x < n+1\}} |\psi(x)| &\leq C\mathbb{E}e^{(\alpha-2\varepsilon)(\log(|AR|+|B|))} \mathbb{E}|B|^\varepsilon |AR|^\varepsilon \\ &= C\mathbb{E}(|AR| + |B|)^{(\alpha-2\varepsilon)} |B|^\varepsilon |AR|^\varepsilon \\ &\leq \tilde{C}\mathbb{E}|A|^{(\alpha-\varepsilon)} |B|^\varepsilon |R|^{(\alpha-\varepsilon)} + \mathbb{E}|A|^\varepsilon |B|^{\alpha-\varepsilon} |R|^\varepsilon < \infty, \end{aligned}$$

because $\mathbb{E}|A|^\varepsilon|B|^{\alpha-\varepsilon} < \infty$ by assumption and

$$\mathbb{E}|A|^{(\alpha-\varepsilon)}|B|^\varepsilon \leq \mathbb{E}|A|^\alpha + \mathbb{E}|A|^\varepsilon|B|^{(\alpha-\varepsilon)}\mathbf{1}_{|B|>|A|}.$$

□

4.4. Perpetuity - second order asymptotics. In this section we study the second order asymptotics of $x^\alpha \mathbb{P}(R > x)$. For that we need more stringent assumptions on the distribution of A . We begin with the following technical Lemma.

Lemma 4.7 *Assume (A-1), (B-1)-(B-3), R and (A, B) are independent, $\tilde{L}(x) := x^\alpha \mathbb{P}(R > x) \in R(0)$. If there exists $\beta > 0$ such that*

$$\limsup_{h \rightarrow 0^+} \sup_{a \in \mathbb{R}} h^{-\beta} \mathbb{P}(a < \log A \leq a + h) < \infty,$$

and $\mathbb{E}A^\gamma < \infty$ for some $\gamma > \max\{\alpha, \alpha^2/\beta\}$, then both functions

$$I_1(x) = e^{\alpha x} \mathbb{P}(\max\{AR, B\} \leq e^x < AR + B)$$

and

$$I_2(x) = e^{\alpha x} \mathbb{P}(AR + B \leq e^x < \max\{AR, B\})$$

are $O(L(e^x))$ as $x \rightarrow \infty$.

Proof. Take $\gamma' \in (\max\{\alpha, \alpha^2/\beta\}, \gamma)$. Then, for any $\delta \in (0, 1)$, we have

$$\begin{aligned} I_1(x) &\leq e^{\alpha x} \left(\mathbb{P}(B > e^x/2) + \mathbb{P}(e^{\delta x} < B \leq e^x/2, AR + B > e^x) + \mathbb{P}(A > e^{\alpha x/\gamma'}) \right. \\ &\quad \left. + \mathbb{P}(B \leq e^{\delta x}, A \leq e^{\alpha x/\gamma'}, AR \leq e^x \leq AR + B) \right) \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned}$$

It is clear that $K_1 = O(L(e^x))$. Furthermore, for any $\delta > 0$ taking $0 < \eta < \alpha\delta/(1+\delta)$ we obtain

$$K_2 \leq e^{\alpha x} \mathbb{P}(ARB > e^{(1+\delta)x}/2) \leq e^{\alpha x} 2^{\alpha-\eta} \frac{\mathbb{E}(ARB)_+^{\alpha-\eta}}{e^{(\alpha-\eta)(1+\delta)x}} = o(e^{-sx}).$$

for some $s > 0$. Moreover, since $1 - \gamma/\gamma' < 0$ we have

$$K_3 \leq e^{\alpha x} \frac{\mathbb{E}A^\gamma}{e^{\alpha\gamma x/\gamma'}} = o(e^{-sx})$$

for some $s > 0$ and so K_2 and K_3 are $O(L(e^x))$ as well. For K_4 define $\lambda(x) = 1 - e^{-(1-\delta)x} \rightarrow 1$ and recall that $\alpha/\gamma' < 1$. Then, by (36),

$$\begin{aligned} K_4 &\leq e^{\alpha x} \mathbb{P}(\lambda(x)e^x < AR \leq e^x, R > \lambda(x)e^{(1-\alpha/\gamma')x}) \\ &= e^{\alpha x} \mathbb{P}(x - \log R + \log \lambda(x) < \log A \leq x - \log R, R > \lambda(x)e^{(1-\alpha/\gamma')x}) \\ &\leq Ce^{\alpha x} (-\log \lambda(x))^\beta \mathbb{P}(R > \lambda(x)e^{(1-\alpha/\gamma')x}) \\ &\sim Ce^{\alpha x} e^{-\beta(1-\delta)x} \frac{\tilde{L}(\lambda(x)e^{(1-\alpha/\gamma')x})}{\lambda(x)^\alpha e^{\alpha(1-\alpha/\gamma')x}}, \end{aligned}$$

which is $O(e^{-sx})$ for some $s > 0$ if there exists $\delta > 0$ such that

$$\frac{\alpha^2}{\gamma'\beta} < 1 - \delta,$$

and this follows by the definition of γ' .

We proceed similarly for I_2 writing

$$I_2(x) \leq e^{\alpha x} \left(\mathbb{P}(B \geq e^x) + \mathbb{P}(AR > e^x, -B > e^{\delta x}) + \mathbb{P}(A > e^{\alpha x/\gamma'}) \right. \\ \left. + \mathbb{P}(-B \leq e^{\delta x}, A \leq e^{\alpha x/\gamma'}, AR + B \leq e^x < AR) \right).$$

Then one can show that there exists $\delta > 0$ small enough that $I_2(x) = O(L(e^x))$.

□

The following Theorem is the main result of this Section.

Theorem 4.8 Assume (A-1)-(A-2) and (B-1)-(B-3). Assume further that there exists $\beta > 0$ such that

$$(36) \quad \limsup_{h \rightarrow 0^+} \sup_{a \in \mathbb{R}} h^{-\beta} \mathbb{P}(a < \log A \leq a + h) < \infty$$

and $\mathbb{E}A^\gamma < \infty$ for some $\gamma > \max\{\alpha, \alpha^2/\beta\}$. If the distribution of Z defined by (25) is strongly non-lattice, then as $x \rightarrow \infty$,

$$x^\alpha \mathbb{P}(R > x) = \frac{\tilde{L}(x)}{\rho} + \frac{\mathbb{E}((AR + B)_+^\alpha - (AR)_+^\alpha - B_+^\alpha)}{\alpha \rho} + O(L(x)) + o(1).$$

Proof. We begin the proof in the same way as in the proof of Theorem 4.2. In view of Theorem 3.1 it remains to show that as $x \rightarrow \infty$

$$\int_{\mathbb{R}} \psi_0(x - z) dH(z) = \frac{1}{\mathbb{E}Z} \int_{\mathbb{R}} \psi_0(t) dt + o(1),$$

where

$$\begin{aligned} \psi_0(x) &= e^{\alpha x} (\mathbb{P}(AR + B > e^x) - \mathbb{P}(AR > e^x) - \mathbb{P}(B > e^x)) \\ &= e^{\alpha x} \mathbb{P}(\max\{AR, B\} \leq e^x < AR + B) - e^{\alpha x} \mathbb{P}(AR + B \leq e^x < \max\{AR, B\}) \\ &\quad - e^{\alpha x} \mathbb{P}(\min\{AR, B\} > e^x) = I_1(x) - I_2(x) - I_3(x). \end{aligned}$$

In the proof of Theorem 4.2 we have already shown that $\int_{\mathbb{R}} I_3(x - z) dH(z)$ converges to $\frac{\mathbb{E} \min\{AR, B\}_+^\alpha}{\alpha \rho}$, which is finite. By the preceding Lemma we know that $I_i(e^x) = O(L(e^x))$, $i = 1, 2$ and this implies that as $x \rightarrow \infty$,

$$\int_{(-\infty, 0]} I_i(x - z) dH(z) = O(L(e^x)), \quad i = 1, 2.$$

Indeed, consider $\int_{(-\infty, 0]} \frac{L(e^{x-z})}{L(e^x)} dH(z)$. For any $\delta > 0$, the integrand is bounded by $ce^{-\delta z}$ for some $c > 1$ by Potter bounds. Combining this with (15) and Lebesgue's Dominated Convergence Theorem we conclude that

$$(37) \quad \int_{(-\infty, 0]} L(e^{x-z}) dH(z) \sim L(e^x) H(0).$$

A better asymptotics of $H(x)$ than (15) is available here: $e^{\alpha x} H(-x) \rightarrow (-\alpha \mathbb{E} \log A)^{-1}$ as $x \rightarrow \infty$, see Kołodziejek [2016a].

Observe that there exists $\beta^* > 0$ such that

$$(38) \quad \limsup_{h \rightarrow 0^+} \sup_{a \geq 0} h^{-\beta^*} \mathbb{P}(a < Z \leq a + h) < \infty.$$

Indeed, let $p = \frac{\alpha + \varepsilon}{\alpha}$, $q = \frac{\alpha + \varepsilon}{\varepsilon}$. Then

$$\mathbb{P}(a < Z \leq a + h) = \mathbb{E}A^\alpha \mathbf{1}_{a < \log A \leq a + h} \leq (\mathbb{E}A^{\alpha + \varepsilon})^{1/p} (\mathbb{P}(a < \log A \leq a + h))^{1/q}.$$

Hence

$$h^{-\beta/q} \mathbb{P}(a < Z \leq a + h) \leq (\mathbb{E}A^{\alpha + \varepsilon})^{1/p} (h^{-\beta} \mathbb{P}(a < \log A \leq a + h))^{1/q}$$

and (38) follows by (36). In view of (18) we have the following easy result for $x > u$ and $d > u$,

$$(39) \quad \begin{aligned} \int_{((x-d)_+, x-u]} e^{\alpha(x-z)} dH(z) &\leq e^{\alpha d} (H(x-u) - H((x-d)_+)) \\ &\leq ce^{\alpha d} \max\{(x-u - (x-d)_+)^{\tilde{\beta}}, x-u - (x-d)_+\} \\ &\leq ce^{\alpha d} \max\{(d-u)^{\tilde{\beta}}, d-u\} \end{aligned}$$

for some $\tilde{\beta} > 0$, where, the first inequality follows from monotonicity of the integrand and the second one by Lemma 2.1.

Moreover, notice that for $0 < \lambda \leq 1$ and all $x > 0$ one has $\log(1+x) \leq \lambda^{-1}x^\lambda$. Let us denote $U = \log \max\{AR, B\}$ and $D = \log(AR+B)$. Then, by (39)

$$\begin{aligned} \int_{(0, \infty)} I_1(x-z) dH(z) &= \mathbb{E} \int_{(0, \infty)} e^{\alpha(x-z)} \mathbf{1}_{\max\{AR, B\} \leq e^{x-z} < AR+B} dH(z) \\ &= \mathbb{E} \int_{(x-D, x-U] \cap (0, \infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{D > U} \\ &\leq c\mathbb{E}(AR+B)^\alpha ((D-U)^{\tilde{\beta}} + (D-U)) \mathbf{1}_{D > U}. \end{aligned}$$

For the first term above we have

$$\begin{aligned} &c\mathbb{E}(AR+B)^\alpha (D-U)^{\tilde{\beta}} \mathbf{1}_{D > U} \\ &\leq c\mathbb{E}(AR+B)^\alpha \left(\log \left(1 + \frac{\min\{AR, B\}}{\max\{AR, B\}} \right) \right)^{\tilde{\beta}} \mathbf{1}_{AR+B > \max\{AR, B\}} \\ &\leq \frac{c}{\lambda^{\tilde{\beta}}} \mathbb{E}(AR+B)^\alpha \frac{\min\{AR, B\}^{\lambda\tilde{\beta}}}{\max\{AR, B\}^{\lambda\tilde{\beta}}} \mathbf{1}_{AR+B > \max\{AR, B\}} \\ &\leq 2^\alpha \frac{c}{\lambda^{\tilde{\beta}}} \mathbb{E} \max\{AR, B\}^{\alpha-\lambda\tilde{\beta}} \min\{AR, B\}^{\lambda\tilde{\beta}} \mathbf{1}_{AR+B > \max\{AR, B\}} \\ &\leq 2^\alpha \frac{c}{\lambda^{\tilde{\beta}}} (\mathbb{E}(AR)^{\alpha-\lambda\tilde{\beta}} B^{\lambda\tilde{\beta}} \mathbf{1}_{\min\{AR, B\}=B > 0} + \mathbb{E} B^{\alpha-\lambda\tilde{\beta}} (AR)^{\lambda\tilde{\beta}} \mathbf{1}_{\min\{AR, B\}=AR > 0}) < \infty \end{aligned}$$

provided $\tilde{\beta}\lambda < \alpha$. An analogous calculation shows that $\mathbb{E}(AR+B)^\alpha (D-U) \mathbf{1}_{D > U} < \infty$ and so $\int_{(x-D, x-U] \cap (0, \infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{D > U}$ is dominated independently of x by an integrable function. Thus, by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{x \rightarrow \infty} \int_{(0, \infty)} I_1(x-z) dH(z) = \mathbb{E} \lim_{x \rightarrow \infty} \int_{(0, x-U]} e^{\alpha(x-z)} \mathbf{1}_{x-z < D} dH(z) \mathbf{1}_{D > U}$$

and for $d > u$ as $x \rightarrow \infty$,

$$e^{\alpha u} \int_{(0, x-u]} e^{\alpha(x-u-z)} \mathbf{1}_{x-u-z < d-u} dH(z) \rightarrow \frac{1}{\mathbb{E}Z} e^{\alpha u} \int_0^\infty e^{\alpha t} \mathbf{1}_{t < d-u} dt,$$

where we have used the Key Renewal Theorem since the integrand is dRi (it has compact support). Thus

$$\lim_{x \rightarrow \infty} \int_{(0, \infty)} I_1(x-z) dH(z) = \frac{\mathbb{E}((AR+B)^\alpha - \max\{AR, B\}^\alpha) \mathbf{1}_{AR+B > \max\{AR, B\}}}{\alpha\rho}.$$

We proceed similarly with I_2 . With $D = \log \max\{AR, B\}$ and $U = \log(AR + B)$, we have

$$\begin{aligned} \int_{(0,\infty)} I_2(x-z) dH(z) &= \mathbb{E} \int_{(0,\infty)} e^{\alpha(x-z)} \mathbf{1}_{AR+B \leq e^{x-z} < \max\{AR, B\}} dH(z) \\ &\leq \mathbb{E} \int_{(x-D, x-U] \cap (0,\infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0} \\ &+ \mathbb{E} \int_{(x-D, \infty) \cap (0,\infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{AR+B \leq 2^{-1} \max\{AR, B\}, \max\{AR, B\} > 0}. \end{aligned}$$

and by (39)

$$\begin{aligned} \mathbb{E} \int_{(x-D, x-U] \cap (0,\infty)} e^{\alpha(x-z)} dH(z) \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0} \\ \leq \mathbb{E} \max\{AR, B\}^\alpha \left((\log \max\{AR, B\} - \log(AR+B)) \right)^{\tilde{\beta}} \\ + (\log \max\{AR, B\} - \log(AR+B)) \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0}. \end{aligned}$$

Again, as before we do calculations for the term with $\tilde{\beta}$. It is bounded by

$$\begin{aligned} c \mathbb{E} \max\{AR, B\}^\alpha \left(\log \left(1 + \frac{-\min\{AR, B\}}{AR+B} \right) \right)^{\tilde{\beta}} \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0} \\ \leq \frac{c}{\lambda^{\tilde{\beta}}} \mathbb{E} \max\{AR, B\}^\alpha \left(\frac{-\min\{AR, B\}}{AR+B} \right)^{\lambda \tilde{\beta}} \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0} \\ \leq 2^{\lambda \tilde{\beta}} \frac{c}{\lambda^{\tilde{\beta}}} \mathbb{E} \max\{AR, B\}^\alpha \left(\frac{|\min\{AR, B\}|}{\max\{AR, B\}} \right)^{\lambda \tilde{\beta}} \mathbf{1}_{\max\{AR, B\} > AR+B \geq 2^{-1} \max\{AR, B\} > 0} \\ \leq 2^{\lambda \tilde{\beta}} \frac{c}{\lambda^{\tilde{\beta}}} \mathbb{E} \max\{AR, B\}^{\alpha - \lambda \tilde{\beta}} |\min\{AR, B\}|^{\lambda \tilde{\beta}} < \infty \end{aligned}$$

as before. Similarly as in (29), the second term equals

$$\begin{aligned} \alpha \mathbb{E} \max\{AR, B\}^\alpha \int_0^\infty e^{-\alpha t} (H(x-D+t) - H(x-D)) dt \mathbf{1}_{AR+B \leq 2^{-1} \max\{AR, B\}, \max\{AR, B\} > 0} \\ \leq c \mathbb{E} \max\{AR, B\}^\alpha \mathbf{1}_{AR+B \leq 2^{-1} \max\{AR, B\}, \max\{AR, B\} > 0}. \end{aligned}$$

Now, since $\min\{AR, B\} \leq 0$ and

$$AR+B = \max\{AR, B\} + \min\{AR, B\} \leq \frac{1}{2} \max\{AR, B\}$$

we have

$$|\min\{AR, B\}| \geq \frac{1}{2} \max\{AR, B\}$$

and

$$\begin{aligned} \mathbb{E} \max\{AR, B\}^\alpha \mathbf{1}_{|\min\{AR, B\}| \geq 2^{-1} \max\{AR, B\} > 0} &\leq \mathbb{E} B^\alpha \mathbf{1}_{B > 0, AR < 0, 1 \leq 2 \frac{|AR|}{B}} + \mathbb{E} (AR)^\alpha \mathbf{1}_{AR > 0, B < 0, 1 \leq 2 \frac{|B|}{AR}} \\ &\leq 2^\eta \left(\mathbb{E} |B|^\alpha \left(\frac{|AR|}{|B|} \right)^\eta + \mathbb{E} |AR|^\alpha \left(\frac{|B|}{|AR|} \right)^\eta \right) \\ &\leq 2^\eta (\mathbb{E} |R|^\eta \mathbb{E} |B|^{\alpha-\eta} A^\eta + \mathbb{E} |R|^{\alpha-\eta} \mathbb{E} A^{\alpha-\eta} |B|^\eta) < \infty. \end{aligned}$$

Similarly as before, Lebesgue's Dominated Convergence Theorem implies that as $x \rightarrow \infty$,

$$\int_{(0,\infty)} I_2(x-z) dH(z) \rightarrow \frac{\mathbb{E}(\max\{AR, B\}^\alpha - (AR+B)_+^\alpha)}{\alpha \mathbb{E} Z}$$

and so as $x \rightarrow \infty$, after straightforward simplification,

$$\int_{\mathbb{R}} \psi_0(x-z) dH(z) = \frac{1}{\alpha\rho} \mathbb{E}((AR+B)_+^\alpha - (AR)_+^\alpha - B_+^\alpha) + O(L(e^x)) + o(1).$$

□

4.5. General A. Now we are going to consider perpetuities with A attaining negative values as well. More precisely, we assume that $\mathbb{P}(A < 0) > 0$, possibly with $\mathbb{P}(A \leq 0) = 1$. Our aim is to reduce the general case to the one already solved: non-negative A . We propose a unified approach to perpetuities, which applies beyond our particular assumptions.

Assume that $\mathbb{E} \log |A| < 0$ and $\mathbb{E} \log^+ |B| < \infty$. Then the stochastic equation $R \stackrel{d}{=} AR+B$ with (A, B) and R independent has a unique solution, or equivalently, that $R_n = A_n R_{n-1} + B_n$, $n \geq 1$, converges in distribution to R for any R_0 independent of $(A_n, B_n)_{n \geq 1}$, where $(A_n, B_n)_{n \geq 1}$ is a sequence of independent copies of the pair (A, B) . For the tail of R we have the following statement.

Theorem 4.9 *Suppose that*

- (sA-1) $\mathbb{P}(A < 0) > 0$, $\mathbb{E} \log |A| < 0$,
- (sA-2) *there exists $\alpha > 0$ such that $\mathbb{E}|A|^\alpha = 1$, $\rho = \mathbb{E}|A|^\alpha \log |A| < \infty$,*
- (sA-3) *the distribution of $\log |A|$ given $|A| > 0$ is non-arithmetic,*
- (sA-4) *there exists $\varepsilon > 0$ such that $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$,*
- (sB-1)

$$\mathbb{P}(B > t) \sim pt^{-\alpha} L(t), \quad \mathbb{P}(B < -t) \sim qt^{-\alpha} L(t), \quad p+q=1,$$

- (sB-2) $\mathbb{E}|B|^\alpha = \infty$.

Then

$$(40) \quad x^\alpha \mathbb{P}(R > x) \sim \frac{\tilde{L}(x)}{2\rho}.$$

Take $R_0 = 0$ and define the filtration $\mathbb{F} = \{\mathcal{F}_n : n \geq 1\}$, where $\mathcal{F}_n = \sigma((A_k, B_k)_{k=1}^n)$. Following [Vervaat, 1979, Lemma 1.2], for any stopping time N (with respect to \mathbb{F}) which is finite with probability one, R satisfies

$$R \stackrel{d}{=} A_1 \dots A_N R + R_N^*, \quad R \text{ and } (A_1 \dots A_N, R_N^*) \text{ are independent,}$$

where $R_n^* = B_1 + A_1 B_2 + \dots + A_1 \dots A_{n-1} B_n$ for $n \geq 1$.

Let $N := \inf\{n : \Pi_n \geq 0\}$. Then, N is a stopping time with respect to \mathbb{F} and N is finite with probability 1. Indeed, if $\mathbb{P}(A \leq 0) = 1$ then $N = 2$. If $\mathbb{P}(A > 0) > 0$ then $N = \infty$ if and only if $A_1 < 0$ and for every $n \geq 2$, $A_n > 0$ which means that for every n

$$(41) \quad \mathbb{P}(N = \infty) \leq \mathbb{P}(A < 0) \mathbb{P}(A > 0)^{n-1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

To conclude we need to prove that Π_N and R_N^* satisfy assumptions of Theorem 4.4. We will now prove that Π_N inherits its properties from A . The following result is strongly inspired by [Goldie, 1991, (9.11)-(9.13)] (see also [Alsmeyer, 2015, Lemma 4.12]). For completeness, we give a proof below.

- Theorem 4.10** (i) *If the law of $\log |A|$ given $A \neq 0$ is non-arithmetic (spread-out), then the law of $\log \Pi_N$ given $\Pi_N > 0$ is non-arithmetic (spread-out),*
- (ii) *If $\mathbb{E}|A|^\alpha = 1$ and $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$ then there exists $\bar{\varepsilon} > 0$ such that $\mathbb{E}\Pi_N^{\alpha+\bar{\varepsilon}} < \infty$,*
 - (iii) *If $\mathbb{E}|A|^\alpha = 1$ then $\mathbb{E}\Pi_N^\alpha = 1$ and $\mathbb{E}\Pi_N^\alpha \log \Pi_N = 2\mathbb{E}|A| \log |A|$.*

Proof. If $P(A \leq 0) = 1$ then $\Pi_N = A_1 A_2$ and the law of $\log \Pi_N$ given $\log \Pi_N > 0$ is $\mathbb{P}_< * \mathbb{P}_<$, where $\mathbb{P}_<$ is the law of $\mathbb{P}_{\log |A| | A < 0}$. $\mathbb{P}_< * \mathbb{P}_<$ is non-arithmetic or spread out respectively if so is $\mathbb{P}_<$. Also the rest of the above statements are clear in this case so for the rest of the proof we assume that $P(A > 0) > 0$.

- (i) Denote by $\mathbb{P}_>$ and $\mathbb{P}_<$ the laws of $\mathbb{P}_{\log A|A>0}$ and $\mathbb{P}_{\log |A||A<0}$, respectively. Set $p = \mathbb{P}(A > 0)$ and $q = \mathbb{P}(A < 0)$. By [Goldie, 1991, (9.11)], we have

$$\mathbb{P}_{\log \Pi_N | \Pi_N > 0} = \frac{1}{\mathbb{P}(\Pi_N > 0)} \left(p\mathbb{P}_> + q^2\mathbb{P}_<^{*2} \sum_{n=0}^{\infty} p^n \mathbb{P}_>^{*n} \right).$$

If $p\mathbb{P}_> + q\mathbb{P}_<$ is spread out then there are $k, l \geq 0$ such that $\mathbb{P}_>^{*k} * \mathbb{P}_<^{*l}$ has a non zero absolutely continuous component. Hence $\mathbb{P}_> * \mathbb{P}_<^{*2}$ is spread out and the mixture of measures, one of which is spread-out is spread-out as well.

If $p\mathbb{P}_> + q\mathbb{P}_<$ is non-arithmetic then the supports of $\mathbb{P}_>$ and $\mathbb{P}_<^{*2}$ generate a dense subgroup of \mathbb{R} (see the argument below [Goldie, 1991, (9.13)]) and so does the support of η . Thus, we conclude that $\mathbb{P}_{\log \Pi_N | \Pi_N > 0}$ is non-arithmetic.

- (ii) Let $\mu_+^{(\varepsilon)} := \mathbb{E}A^{\alpha+\varepsilon} \mathbf{1}_{A \geq 0}$. Since the function $\varepsilon \mapsto \mu_+^{(\varepsilon)}$ is continuous and $\mu_+^{(0)} < 1$, then there exists $\varepsilon_1 > 0$ such that $\mu_+^{(\varepsilon_1)} < 1$.

Then, we have

$$\begin{aligned} \mathbb{E}\Pi_N^{\alpha+\varepsilon_1} &= \mathbb{E}A_1^{\alpha+\varepsilon_1} \mathbf{1}_{A_1 \geq 0} + \sum_{n=2}^{\infty} \mathbb{E}\Pi_n^{\alpha+\varepsilon_1} \mathbf{1}_{A_1 < 0, A_2 > 0, \dots, A_{n-1} > 0, A_n \leq 0} \\ &= \mu_+^{(\varepsilon_1)} + (\mathbb{E}|A|^{\alpha+\varepsilon_1} \mathbf{1}_{A < 0})^2 \sum_{n=2}^{\infty} (\mu_+^{(\varepsilon_1)})^{n-2} < \infty. \end{aligned}$$

- (iii) Define a measure \mathbb{Q}_n on (Ω, \mathcal{F}_n) by

$$\mathbb{Q}_n(S) := \mathbb{E}|\Pi_n|^\alpha \mathbf{1}_S, \quad S \in \mathcal{F}_n, \quad n \geq 0.$$

Let \mathcal{F}_∞ be the smallest σ -field containing all \mathcal{F}_n . The sequence of measures \mathbb{Q}_n is consistent, thus by Kolmogorov theorem there exists a unique measure \mathbb{Q} on \mathcal{F}_∞ such that $\mathbb{Q}(S) = \mathbb{Q}_n(S)$ for $S \in \mathcal{F}_n$. Note that $(A_n)_{n \geq 1}$ are i.i.d. also under \mathbb{Q} . We have

$$\mu_+ := \mathbb{Q}(N = 1) = \mathbb{Q}(A_1 \geq 0) = \mathbb{E}|A|^\alpha \mathbf{1}_{A > 0} = \mu_+^{(0)}$$

and for any $k > 1$,

$$\mathbb{Q}(N = k) = \mathbb{Q}(A_1 < 0, A_2 > 0, \dots, A_{k-1} > 0, A_k \leq 0) = (1 - \mu_+)^2 \mu_+^{k-2}.$$

Hence $\mathbb{E}_\mathbb{Q} N = 2$, where $\mathbb{E}_\mathbb{Q}$ is the expectation with respect to \mathbb{Q} .

Since $\mathcal{F}_N \subset \mathcal{F}_\infty$, for any $S \in \mathcal{F}_N$ we have

$$\begin{aligned} \mathbb{Q}(S) &= \sum_{n=1}^{\infty} \mathbb{Q}(S \cap \{N = n\}) = \sum_{n=1}^{\infty} \mathbb{E}|\Pi_n|^\alpha \mathbf{1}_{S \cap \{N=n\}} \\ &= \sum_{n=1}^{\infty} \mathbb{E}\Pi_n^\alpha \mathbf{1}_{S \cap \{N=n\}} = \mathbb{E}\Pi_N^\alpha \mathbf{1}_S. \end{aligned}$$

Putting $S = \Omega$ we obtain that $\mathbb{E}\Pi_N^\alpha = 1$. Further, since Π_N is \mathcal{F}_N measurable, we have

$$\mathbb{E}\Pi_N^\alpha \log \Pi_N = \mathbb{E}_\mathbb{Q} \log \Pi_N = \mathbb{E}_\mathbb{Q} \left(\sum_{n=1}^N \log |A_n| \right) = \mathbb{E}_\mathbb{Q} N \cdot \mathbb{E}_\mathbb{Q} \log |A_1| = 2\mathbb{E}|A|^\alpha \log |A|,$$

where the Wald's identity was used.

□

We are going to prove that the tails of R_N^* behave like $\mathbb{P}(|B| > x)$. Let now $\mathbb{P}(A > 0) > 0$ and $A_+ = A \mathbf{1}_{A \geq 0}$, $A_- = -A \mathbf{1}_{A < 0}$.

Theorem 4.11

(i)

$$R_N^* \stackrel{d}{=} (-A_-)S + B,$$

where S and (A_-, B) are independent and S satisfies the following stochastic equation

$$(42) \quad S \stackrel{d}{=} A_+ S + B, \quad S \text{ and } (A_+, B) \text{ are independent.}$$

(ii) Assume additionally that

$$\mathbb{P}(B > t) \sim pt^{-\alpha}L(t), \quad \mathbb{P}(B < -t) \sim qt^{-\alpha}L(t), \quad p + q = 1.$$

and $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. If $\mu_+ = \mathbb{E}A^\alpha \mathbf{1}_{A>0} < 1$, then

$$(43) \quad \mathbb{P}(S > t) \sim \frac{1}{1 - \mu_+} \mathbb{P}(B > t), \quad \mathbb{P}(S < -t) \sim \frac{1}{1 - \mu_+} \mathbb{P}(B < -t),$$

and

$$(44) \quad \mathbb{P}(R_N^* > t) \sim \mathbb{P}(|B| > t) \sim \mathbb{P}(R_N^* < -t).$$

Proof. (i) Since $\{N \geq k\} = \{A_1 < 0, A_2 > 0, \dots, A_{k-1} > 0\}$ for $k \geq 2$ we have

$$R_N^* = \sum_{k=1}^{\infty} \mathbf{1}_{N \geq k} \Pi_{k-1} B_k = B_1 - (A_1)_- \left(\sum_{k=2}^{\infty} (A_2)_+ \cdots (A_{k-1})_+ B_k \right).$$

Let us denote the expression in brackets by S . Then, S is independent of $((A_1)_-, B_1)$ and satisfies (42).

(ii) Tail asymptotic of S follow from the application of [Grey, 1994, Theorem 3] to $(M, Q, R) = (A_+, B, S)$. We have $\mathbb{E}|M|^\alpha = \mathbb{E}A^\alpha \mathbf{1}_{A>0} < 1$ and $\mathbb{E}|M|^{\alpha+\varepsilon} \leq \mathbb{E}|A|^{\alpha+\varepsilon} < \infty$ by the assumption.

Tail asymptotics of R_N^* then follow from [Grey, 1994, Lemma 4], since $R_N^* \stackrel{d}{=} B + A^- S$. Here $(M, Q, Y) = (A_-, B, S)$ and $\mathbb{E}|M|^\alpha = \mathbb{E}A^\alpha \mathbf{1}_{A<0} < 1$ and $\mathbb{E}|M|^{\alpha+\varepsilon}$ is finite as above. One easily checks that $\mathbb{P}(R_N^* > t) \sim \mathbb{P}(|B| > t)$. To obtain $\mathbb{P}(R_N^* < -t) \sim \mathbb{P}(|B| > t)$ we apply the above argument to $-R \stackrel{d}{=} A(-R) - B$. □

5. PROOF OF THEOREM 3.1

1. Proof of (22)

First we prove that

$$(45) \quad \lim_{x \rightarrow \infty} \tilde{L}(e^x)^{-1} \int_{(-\infty, 0]} e^{\alpha(x-z)} \mathbb{E}g(e^{z-x}B) dH(z) = 0$$

Since g is bounded as its support is contained in $[1, \infty)$, there exists a constant c such that $g(x) \leq c \mathbf{1}_{x>1-\varepsilon}$ for any $\varepsilon > 0$. Thus, $e^{\alpha(x-z)} \mathbb{E}g(e^{z-x}B) \leq ce^{\alpha(x-z)} \mathbb{P}(B > (1-\varepsilon)e^{x-z}) = cL((1-\varepsilon)e^{x-z})$ and therefore

$$\int_{(-\infty, 0]} e^{\alpha(x-z)} \mathbb{E}g(e^{z-x}B) dH(z) \leq c \int_{(-\infty, 0]} L((1-\varepsilon)e^{x-z}) dH(z) \sim cL(e^x)H(0) = o(\tilde{L}(e^x)).$$

by (37).

For the main part we have

$$\begin{aligned}
\int_{(0,\infty)} \psi_B(x-z) dH(z) &= \int_{(0,\infty)} e^{\alpha(x-z)} \mathbb{E}g(e^{-(x-z)}B) \mathbf{1}_{\{B > e^x\}} dH(z) \\
&+ \int_{(0,\infty)} e^{\alpha(x-z)} \mathbb{E}g(e^{-(x-z)}B) \mathbf{1}_{\{0 < B \leq e^x\}} dH(z) \\
&\leq c \int_{(0,\infty)} e^{\alpha(x-z)} \mathbb{P}(B > (1-\varepsilon)e^{x-z}, B > e^x) dH(z) \\
&+ \int_{(0,\infty)} e^{\alpha(x-z)} \mathbb{E}g(e^{-(x-z)}B) \mathbf{1}_{\{0 < B \leq e^x\}} dH(z) \\
&= cL(e^x) \int_{(0,\infty)} e^{-\alpha z} dH(z) + \mathbb{E} \int_{(0,\infty)} e^{\alpha(x-z)} g(e^{-(x-z)}B) dH(z) \mathbf{1}_{0 < B \leq e^x}.
\end{aligned}$$

Since $\int_{(0,\infty)} e^{-\alpha z} dH(z)$ is finite, the first term is $O(L(e^x))$. We are going to compare the second term above with

$$K(x) := \mathbb{E} \int_0^\infty e^{\alpha(x-z)} g(e^{-(x-z)}B) d\frac{z}{\mathbb{E}Z} \mathbf{1}_{0 < B \leq e^x}.$$

Changing variables $r = e^{-(x-z)}B$, we have

$$\begin{aligned}
K(x) &= (\mathbb{E}Z)^{-1} \mathbb{E}B_+^\alpha \mathbf{1}_{B \leq e^x} \int_{e^{-x}B}^\infty r^{-\alpha} g(r) \frac{dr}{r} \\
&= (\mathbb{E}Z)^{-1} \mathbb{E}B_+^\alpha \mathbf{1}_{B \leq e^x} \int_0^\infty g(r) r^{-\alpha-1} dr,
\end{aligned}$$

by the fact that $\text{supp } g \subset [1, \infty)$. But

$$\mathbb{E}B_+^\alpha \mathbf{1}_{B \leq e^x} = \alpha \tilde{L}(e^x).$$

Hence

$$K(x) = \alpha (\mathbb{E}Z)^{-1} \tilde{L}(e^x) \int_0^\infty g(r) r^{-\alpha-1} dr.$$

It remains to prove that

$$(46) \quad \mathbb{E} \int_{(0,\infty)} e^{\alpha(x-z)} g(e^{-(x-z)}B) d\left(H(z) - \frac{z}{\mathbb{E}Z}\right) \mathbf{1}_{0 < B \leq e^x} = o(\tilde{L}(e^x))$$

For (46) let

$$R(z) = H(z) - \frac{z}{\mathbb{E}Z}.$$

Since $g(1) = 0$ and $\lim_{z \rightarrow \infty} e^{-\alpha z} R(z) = 0$, after integrating by parts we arrive at

$$\begin{aligned}
&\mathbb{E} \int_{(x-\log B, \infty)} e^{\alpha(x-z)} g(e^{-(x-z)}B) dR(z) \mathbf{1}_{0 < B \leq e^x} \\
&= -\mathbb{E} \int_{x-\log B}^\infty \frac{d}{dz} \left(e^{\alpha(x-z)} g(e^{-(x-z)}B) \right) R(z) dz \mathbf{1}_{0 < B \leq e^x} \\
&= -\mathbb{E} B^\alpha \int_0^\infty \frac{d}{dt} \left(e^{-\alpha t} g(e^t) \right) R(t+x-\log B) dt \mathbf{1}_{0 < B \leq e^x},
\end{aligned}$$

where $t = z - x + \log B$. Moreover, notice that

$$\int_0^\infty \frac{d}{dt} \left(e^{-\alpha t} g(e^t) \right) dt = 0$$

and so

$$\begin{aligned} & \mathbb{E} B^\alpha \int_0^\infty \frac{d}{dt} \left(e^{-\alpha t} g(e^t) \right) R(t+x-\log B) dt \mathbf{1}_{0 < B \leq e^x} \\ &= \mathbb{E} B^\alpha \int_0^\infty \frac{d}{dt} \left(e^{-\alpha t} g(e^t) \right) (R(t+x-\log B) - R(x-\log B)) dt \mathbf{1}_{0 < B \leq e^x} \end{aligned}$$

By the fact that g and g' are bounded, we have for a constant $C = C(g)$

$$\left| \frac{d}{du} (e^{-\alpha t} g(e^t)) \right| \leq C e^{-\alpha t},$$

so it amounts to estimate

$$(47) \quad \mathbb{E} B^\alpha \mathbf{1}_{0 < B \leq e^x} \int_0^\infty e^{-\alpha t} |R(t+x-\log B) - R(x-\log B)| dt$$

Define

$$J(x) := \frac{\mathbb{E} B^\alpha \mathbf{1}_{0 < B \leq e^x} \int_0^\infty e^{-\alpha t} |R(t+x-\log B) - R(x-\log B)| dt}{\mathbb{E} B^\alpha \mathbf{1}_{0 < B \leq e^x}}.$$

We will show that $J(x) \rightarrow 0$, and since the denominator equals $\alpha \tilde{L}(e^x)$ this will be the end of the proof.

Define the law of C_x by

$$\mathbb{P}(C_x \in \cdot) = \frac{\mathbb{E} B^\alpha I(0 < B \leq e^x, B \in \cdot)}{\mathbb{E} B^\alpha I(0 < B \leq e^x)}.$$

Note that $\mathbb{P}(0 < C_x \leq e^x) = 1$. Thus, $J(x)$ may be rewritten as

$$\mathbb{E} \int_0^\infty e^{-\alpha t} |R(t+x-\log C_x) - R(x-\log C_x)| dt.$$

Since for any positive x and t , $|R(t+x) - R(x)| = |H(x+t) - H(x) - \frac{t}{\mathbb{E} Z}| \leq ct + b$ for some $c, b > 0$, we have

$$\lim_{x \rightarrow \infty} J(x) = \int_0^\infty e^{-\alpha t} \lim_{x \rightarrow \infty} \mathbb{E} |R(t+x-\log C_x) - R(x-\log C_x)| dt.$$

Moreover, $x - C_x$ converges to infinity in probability, as $x \rightarrow \infty$. For any $N > 0$ we have

$$\mathbb{P}(x - \log C_x \geq N) = \mathbb{P}(C_x \leq e^{x-N}) = \frac{\mathbb{E} B_+^\alpha I(B \leq e^{x-N})}{\mathbb{E} B_+^\alpha I(B \leq e^x)} = \frac{\tilde{L}(e^{x-N})}{\tilde{L}(e^x)} \rightarrow 1,$$

because \tilde{L} is slowly varying. Since, $|R(t+x) - R(x)| \rightarrow 0$ as $x \rightarrow \infty$, we infer that

$$(48) \quad |(R(t+x-\log C_x) - R(x-\log C_x))|$$

converges to 0 in probability, as $x \rightarrow \infty$. But (48) is bounded, thus the convergence holds also in L_1 and we may finally conclude that

$$\lim_{x \rightarrow \infty} J(x) = 0,$$

which completes the proof.

2. Proof of (21) We have

$$\begin{aligned} \int_{\mathbb{R}} L(e^{x-z}) dH(z) &= \int_{(-\infty, 0]} L(e^{x-z}) dH(z) + \int_{(0, \infty)} e^{\alpha(x-z)} \mathbb{P}(B > e^x) dH(z) \\ &\quad + \int_{(0, \infty)} e^{\alpha(x-z)} \mathbb{P}(e^x \geq B > e^{x-z}) dH(z). \end{aligned}$$

We already know that the first term is asymptotically equivalent to $L(e^x)H(0)$. The second term equals $L(e^x) \int_0^\infty e^{-\alpha z} dH(z)$ and the integral is convergent, thus it is of the same order as the first one.

The main contribution comes from the third term, which is equal to

$$(\mathbb{E}Z)^{-1} \int_0^\infty e^{\alpha(x-z)} \mathbb{P}(x \geq B > e^{x-z}) dz + \mathbb{E} \int_{(x-\log B, \infty)} e^{\alpha(x-z)} dR(z) \mathbf{1}_{0 < B \leq e^x} = K_1(x) + K_2(x),$$

where, as before, $R(z) = H(z) - \frac{z}{\mathbb{E}Z}$. Analogously as in the previous step, we have

$$K_1(x) = \frac{1}{\mathbb{E}Z} \tilde{L}(e^x)$$

and after integrating by parts and changing the variable $t = z - x + \log B$,

$$|K_2(x)| \leq \mathbb{E}B^\alpha I(0 < B \leq e^x) \alpha \int_0^\infty e^{-\alpha t} |(R(t+x-\log B) - R(x-\log B))| dt = \alpha^2 J(x) \tilde{L}(e^x).$$

From the proof of previous case, we already know that $\lim_{x \rightarrow \infty} J(x) = 0$ and (21) follows.

3. Proof of (23) Since $\mathbb{E}e^{\varepsilon Z} < \infty$, we get that $e^{\varepsilon x} \mathbb{P}(Z > x) \rightarrow 0$, thus (16) holds. Let $R(x) = H(x) - \frac{x}{\mu}$ and let $C = \frac{\mathbb{E}Z^2}{2\mathbb{E}Z}$. By (16), there exists $r > 0$ such that $R(x) - C = o(e^{-rx})$. Thus $|R(x) - C| \leq K e^{-rx}$ for all $x > 0$ and some finite K . Splitting $\int_{\mathbb{R}} L(e^{x-z}) dH(z)$ as in the previous case, it remains to show that $K_2(x) = O(L(e^x))$. We have

$$\begin{aligned} |K_2(x)| &\leq \alpha \mathbb{E}B^\alpha I(0 < B \leq e^x) \int_0^\infty e^{-\alpha t} |(R(t+x-\log B) - C)| dt \\ &\quad + \alpha \mathbb{E}B^\alpha I(0 < B \leq e^x) |R(x-\log B) - C| \\ &\leq K \left(1 + \frac{\alpha}{\alpha + r}\right) e^{-rx} \mathbb{E}B_+^{\alpha+r} \mathbf{1}_{B \leq e^x} = O(L(e^x)) \end{aligned}$$

by Proposition 4.1.

6. APPENDIX

Suppose that $\mathbb{E}|B|^\beta < \infty$ for any $\beta < \alpha$ and that there is $\varepsilon > 0$ such that $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$. Then by Hölder inequality we may conclude that for every $\eta < \alpha$, $\mathbb{E}|B|^{\alpha-\eta}|A|^\eta < \infty$. However, if the tail of B exhibits some more regularity, a weaker condition implies the same conclusion.

Suppose that there is a locally bounded function L_1 such that

$$(49) \quad \mathbb{P}(|B| > t) \leq L_1(t) t^{-\alpha}$$

and L_1 has the following property: for every $\delta > 0$ there are $d, C > 0$ such that

$$(50) \quad L_1(ts) \leq CL_1(t)s^\delta, \quad \text{whenever } t \geq d, s \geq 1.$$

We consider

$$W(t) = \max\{\sup_{w \leq t} L_1(w), \log t\}.$$

Then W is increasing and satisfies (50) with possibly slightly different C and d .

Lemma 6.1 *Assume that (49) is satisfied, $\eta < \alpha$, $D > \frac{2\alpha}{\eta}$ and*

$$\mathbb{E}|A|^\alpha W(|A|)^D < \infty.$$

Then

$$\mathbb{E}|B|^{\alpha-\eta}|A|^\eta < \infty.$$

Proof. Since $\mathbb{E}|A|^\alpha < \infty$ and $\mathbb{E}|B|^{\alpha-\eta} < \infty$, it is enough to prove that for a fixed C_0

$$\mathbb{E}|B|^{\alpha-\eta}|A|^\eta \mathbf{1}_{|B|>|A|} \mathbf{1}_{|A|\geq C_0} < \infty.$$

We choose γ close to $1 - \frac{\eta}{\alpha}$ such that

$$1 - \frac{\eta}{\alpha} < \gamma < 1 \quad \text{and} \quad D(1 - \gamma) > 2.$$

Let $\xi > 0$ be such that

$$2 + \xi < D(1 - \gamma)$$

and $\beta > 0$ such that

$$-1 - \xi < \beta(\alpha - \eta - \gamma\alpha) < -1.$$

Notice that with our choice of γ ,

$$\alpha - \eta - \gamma\alpha < 0$$

so the latter may be done. Finally, we choose $\delta > 0$ so small that

$$\beta(\alpha - \eta - \gamma\alpha + \gamma\delta) < -1.$$

Now we fix $k_0 \geq 1$ such that (50) holds for W with δ and $t \geq e^{k_0}$. For $m \geq k$ consider the sets

$$S_{k,m} = \{e^k W(e^k)^\beta < |A| \leq e^{k+1} W(e^{k+1})^\beta, e^m W(e^m)^\beta < |B| \leq e^{m+1} W(e^{m+1})^\beta\}$$

Let $C_0 = e^{k_0} W(e^{k_0})^\beta$. Then, for sufficiently large k_0 ,

$$\mathbb{E}|B|^{\alpha-\eta}|A|^\eta \mathbf{1}_{|B|>|A|} \mathbf{1}_{|A|\geq C_0} \leq C \sum_{k \geq k_0} \sum_{m \geq k} e^{m(\alpha-\eta)+k\eta} W(e^m)^{\beta(\alpha-\eta)} W(e^k)^{\beta\eta} \mathbb{P}(S_{k,m}).$$

but

$$\mathbb{P}(S_{k,m}) \leq \mathbb{P}(|B| > e^m W(e^m)^\beta)^\gamma \mathbb{P}(|A| > e^k W(e^k)^\beta)^{1-\gamma}$$

and

$$\begin{aligned} \mathbb{P}(|B| > e^m W(e^m)^\beta) &\leq e^{-\alpha m} W(e^m)^{-\alpha\beta} L_1(e^m W(e^m)^\beta) \\ &\leq C e^{-\alpha m} W(e^m)^{-\alpha\beta+\beta\delta} L_1(e^m) \\ &\leq C e^{-\alpha m} W(e^m)^{1-\alpha\beta+\beta\delta}. \end{aligned}$$

Further,

$$\begin{aligned} \mathbb{P}(|A| > e^k W(e^k)^\beta) &\leq \mathbb{E}|A|^\alpha W(|A|)^D e^{-\alpha k} W(e^k)^{-\alpha\beta} W(e^k W(e^k)^\beta)^{-D} \\ &\leq C e^{-\alpha k} W(e^k)^{-D-\alpha\beta}, \end{aligned}$$

because W is increasing and $W(e^k) \geq 1$. Therefore, we have

$$\begin{aligned} \mathbb{E}|B|^{\alpha-\eta}|A|^\eta \mathbf{1}_{|B|>|A|} \mathbf{1}_{|A|\geq C_0} \\ \leq C \sum_{k \geq k_0} \sum_{m \geq k} e^{(m-k)(\alpha-\eta-\alpha\gamma)} W(e^m)^{\beta(\alpha-\eta)+\gamma(1-\alpha\beta+\beta\delta)} W(e^k)^{\beta\eta-\alpha\beta(1-\gamma)-D(1-\gamma)}. \end{aligned}$$

Due to the choice of γ, β and δ

$$\beta(\alpha - \eta) + \gamma(1 - \alpha\beta + \beta\delta) = \beta(\alpha - \eta - \gamma\alpha + \gamma\delta) + \gamma < 0$$

and

$$\sum_{m \geq k} e^{(m-k)(\alpha-\eta-\alpha\gamma)} = \sum_{m \geq 0} e^{m(\alpha-\eta-\alpha\gamma)} < \infty.$$

Hence

$$\begin{aligned}
\mathbb{E}|B|^{\alpha-\eta}|A|^{\eta}\mathbf{1}_{|B|>|A|}\mathbf{1}_{|A|\geq C_0} \\
\leq C \sum_{k\geq k_0} W(e^k)^{\beta(\eta-\alpha+\alpha\gamma)-D(1-\gamma)} \\
\leq C \sum_{k\geq k_0} W(e^k)^{1+\xi-D(1-\gamma)} < \infty,
\end{aligned}$$

because

$$1 + \xi - D(1 - \gamma) < -1 \quad \text{and} \quad W(e^k) \geq k.$$

□

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